Rectifiable measures: a local-to-global adventure tailored for Carnot groups

Gioacchino Antonelli
Scuola Normale Superiore, Pisa

Joint work with A. Verlo
Based on the arXiv preprint:
“Rectifiable measures in Carnot groups: Structure theory”

February, 11 2021
Def. $E \subseteq \mathbb{R}^n$ is $m$-rectifiable if
\[ \exists \varphi_i : A_i \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n \text{ Lipschitz} \]
\[ \mathcal{H}^m (E \setminus \bigcup_{i=1}^{\infty} \varphi_i (A_i)) = 0 \]

Equivalent building blocks

- Entire Lipschitz images ($A_i = \mathbb{R}^m$)
- $m$-dimensional $C^1$-surfaces $\Sigma_i$
- m-dimensional Lipschitz graphs $\Gamma_i$

\[ \exists \pi \exists \alpha \forall p \in \Gamma \quad \Gamma \cap C(p, \pi, \alpha) = \Gamma \]

\[ C(p, \pi, \alpha) := p + \{ y \in \mathbb{R}^n : |\pi^\perp y| \leq \alpha |\pi y| \} \]
**RECTIFIABILITY OF MEASURES \((\mathbb{R}^n)\)**

**Def.** \(\phi\) Radon measure on \(\mathbb{R}^n\) is \(m\)-rectifiable if
- \(\phi << \mathcal{H}^m\)
- \(\phi(1_{\mathbb{R}^n \setminus E}) = 0\) for \(E \subseteq \mathbb{R}^n\) \(m\)-rect.

**Def.** \(\phi\) \(R.m.\) on \(\mathbb{R}^n\), \(x \in \mathbb{R}^n\)
- \(\Theta^m_*(\phi, x) := \lim_{r \to 0} \frac{\phi(B(x, r))}{r^m}\)
- \(\Theta^m_{*, x}(\phi, x) := \lim_{r \to 0} \frac{\phi(B(x, r))}{r^m}\)

\[0 < \Theta^m_*(\phi, x) \leq \Theta^m_{*, x}(\phi, x) < +\infty\] \((\text{ID})^m\)

**Def.** \(\phi\) \(R.m.\) on \(\mathbb{R}^n\), \((\text{ID})^m\) holds.
- \(\text{Tan}_m(\phi, x) := \{\text{w-lim of } f_i^{-m}(T_{x, r_i}) \# \phi, r_i \downarrow 0\}\)
- \((T_{x, r_i})_\# \phi(A) := \phi(r_i A + x)\)
**Theorem [Marstrand-Mattila]**

\( \phi \) R.m. on \( \mathbb{R}^n \), \((\mathcal{D})_m\) holds for \( \phi \)-a.e. \( x \in \mathbb{R}^n \)

TFAE:

(i) \( \phi \) is \( m \)-rectifiable

(ii) \( \text{Tan}_m(\phi, x) \subseteq \{ \lambda \mathcal{H}^m \mid V : \lambda > 0, \forall m \text{-dim. vector subspace in } \mathbb{R}^n \} \) for \( \phi \)-a.e. \( x \in \mathbb{R}^n \)

**Proof.**

(i) \( \Rightarrow \) (ii)

\( \phi = \varphi \mathcal{H}^m \perp \Sigma \)

\( \text{Tan}_m(\phi, x) = \{ c_x \mathcal{H}^m \mid c_{\Sigma} T_x \leq \} \)

(ii) \( \Rightarrow \) (i) Quite difficult 😞
**Def** A Carnot group \((G, \cdot)\)

- simply connected
- nilpotent Lie group
- with \( g = V_1 \oplus \ldots \oplus V_s \) s.t.

\[
\forall 1 \leq i \leq s, \quad [V_i, V_i] = V_{i+1} \quad \text{and} \quad V_s \neq \{0\} \quad \text{and} \quad [V_i, V_s] = \{0\}.
\]

**Dilations**

\[
\forall \lambda > 0, \quad S_\lambda |_{V_i} = \lambda^2 \text{id} |_{V_i}, \quad \forall 1 \leq i \leq s
\]

**Identification**

\(
\{X_1, \ldots, X_m, X_{m+1}, \ldots, X_n\}\) adapted basis of \(g\)

\[
(\mathbb{R}^n, \cdot) \xleftrightarrow{\text{BCH}}\quad (G, \cdot) \quad \text{exp}(x_1X_1 + \ldots + x_nX_n)
\]
**Def** A distance $d$ on $G$ is

- **homogeneous** if $d(\delta a g_1, \delta a g_2) = \text{Ad}(g_1, g_2)$
- **left-invariant** if $d(gh_1, gh_2) = d(h_1, h_2)$

Denote $d(g, e) = 1g1 := d(g, e)$

**CC-distances**

$$d(g_1, g_2) = \inf \left\{ \int_0^1 \| \dot{\gamma}(t) \| : \gamma(0) = g_1, \gamma(1) = g_2, \forall t \right\}$$

**Def** $W$ subgroup of $G$ is homogeneous if it is closed under $\delta a$

"Flat" subspaces $\equiv$ hom. subgroups

- $W$ hom. sub. $\rightarrow W = W_1 \oplus \ldots \oplus W_s$
AN EXAMPLE: \( H^1 \)

**Rank** = 2  
**Dimension** = 3  
**Step** = 2

\[
H^1 = \text{span} \{X_1, X_2, X_3\} = \text{span} \{X_1, X_2\} = X_3
\]

\[
X_1 = \partial x_1 - \frac{x_2}{2} \partial x_3
\]

\[
X_2 = \partial x_2 + \frac{x_1}{2} \partial x_3
\]

---

**Osservazioni generali (6)**

- Any two hom. left. inv. d. are equivalent
- \( w = w_1 \oplus \ldots \oplus w_s \)  
  \( \dim_+ W = \sum_{i=1}^s i \dim w_i \)
- (Quotients) of C.g.

Infinitesimal models of sub-Riemannian geom.
**Def.** \( E = (G_d) \) is \( m \)-rectifiable if

\[
\exists \text{ "m-dimensional } \Gamma_i \text{"} \\
\mathcal{H}^m (E \setminus \bigcup_{i=1}^{+\infty} \Gamma_i) = 0
\]

"Different" building blocks

- \( \Gamma = f(A) \quad f: A \subseteq \mathbb{R}^m \rightarrow G \text{ Lip.} \)

**Thm.** [Ambrosio - Kirchheim; Magnani:]

\( \mathcal{H}^1 \) is not \( m \)-rectifiable \( \forall m \geq 2 \)

- \( \Gamma = C^1_H \)-hypersurface \( \quad (m = \dim_H G - 1) \)

\( \Rightarrow \) De Giorgi's rectifiability

[Franchi - Serapioni - Serra Cassano; Marchi; Le Donne - Moisala...]

\( \Rightarrow \) Difficult to adapt to low dimensions

\[\text{RECTIFIABILITY (CARNOT GROUPS)}(4)\]
**Def.** \( W \) and \( IL \) hom. subgroups are complementary if

- \( W \cap IL = \{ e \} \)
- \( G = W \cdot IL \quad [g = P_w g \cdot P_{IL} g] \)

\[ \Gamma = \"i.lip. graphs\" \]

\( \varphi: A \leq_W W \rightarrow IL \), \( \Gamma = \{ a \cdot \varphi(a) : a \in A \} \) s.t.

\[ \forall W \exists \alpha \forall p \in \Gamma \quad \Gamma \cap C(p,W,\alpha) = \Gamma \;
\]

\[ C(p,W,\alpha) = p \cdot \{ y \in G : |P_{IL} y| \leq \alpha |P_w y| \} \]

\( i.lip. \) constant \( \implies \inf = \alpha \)
PAULS’ RECTIFIABILITY & SUB-RIEMANNIAN MANIFOLDS
[Pauls, Cole-Pauls; Bigolin-Vittone; Fässler-Di Donato-Orponen...]
[Le Donne-Young; Antonelli-Le Donne...]

WEAK NOTION OF RECTIFIABILITY WITH CONES
[Don-Le Donne-Moiseal-Vittone]

RADENÁCHER (\textit{Lip} \leftrightarrow C^1) & RELATIONS BETWEEN VARIOUS NOTIONS
[Franchi-Serapioni-Sera Cassano; Kennedy-Maiale-Magnani; Vittone
Hajšiš; Cerzoli-Serra Cassano; Julia-Niculsi Gda-Vittone...]

REGULARITY OF PARAMETRIZING FUNCTIONS OF $C^1_H$-SURFACES
[Hogmomij Ambrosio-Sera Cassano-Vittone; Bigolin Serra Cassano;
Bigolin-Ceraveva-Sera Cassano; Kozhevnikov, Di Donato; Cornij Antenelli-Di Donato-Don;
Le Donne; Antonelli-Di Donato-Don...]

OTHER NOTIONS & RELATIONS...
Def. [Merlo]

\( \phi \) R.m. on \((\mathbb{G}, d)\).
- \( \phi \) is \( \mathcal{D}_m \)-rectifiable if for \( \phi \)-a.e. \( x \in \mathbb{G} \)
  (i) \( 0 < \Theta^m_\phi(\phi, x) \leq \Theta^m_\phi(\phi, x) < +\infty \) (TD\(m) \)
  (ii) \( \Gamma^m_\phi(\phi, x) \subseteq \{ \lambda \mathcal{H}^m \cap W(x) : \lambda > 0 \}
  \quad \text{is hom. sub. of } \mathbb{G}, \dim_H W(x) = m \}

- \( \phi \) is \( \mathcal{D}_m^* \)-rectifiable if for \( \phi \)-a.e. \( x \in \mathbb{G} \)
  (i) above holds
  (ii) \( \Gamma^m_\phi(\phi, x) \subseteq \{ \lambda \mathcal{H}^m \cap W : \lambda > 0 \}
  \quad \text{W is hom. subgroup of } \mathbb{G}, \dim_H W = m \}

Recall:

\[ \Gamma^m_\phi(\phi, x) := \{ \text{w-lim } r^{-m}(T_{x, r_i})_\# \phi, \ r_i \downarrow 0 \}
\]

\[ (T_{x, r_i})_\# \phi(A) := \phi(\chi \cdot S_{r_i} A) \]
**Thm [A. Merlo]**

\( \phi \) \( m \)-rectifiable measures on \((G,d)\).

\( \Phi(x) \) is complemented for \( \phi \)-a.e. \( x \in G \).

Then, \( \forall \alpha > 0, \exists \Gamma_i \) \( \text{Lip} \) graphs with \( \text{Lip}_{\Gamma_i} \leq \alpha \) s.t.

\[ \phi(G \setminus \bigcup_{i=1}^{\infty} \Gamma_i) = 0 \]

**Proof**

1. "Wlog" \( \Phi \equiv \mathcal{H}^m \Sigma \)

2. From "near measures" to "near sets".

\[ F_{x_1} (\mathcal{H}^m \Sigma, \Theta \mathcal{H}^m L \omega(x)) \leq \delta \]

\[ \Sigma \cap B(x_1, r) \subseteq B(x, W(x) \omega(\delta) r) \]

3. \( \delta \) can be taken \( \forall \) 0 at smaller scales. Then

\[ \forall \delta \exists r \Sigma \cap B(x_1, r) \subseteq C(x, W(x), \delta) \]
**Theorem (A. - Merlo, WIPJ)**

If \( \Gamma \subseteq \Gamma \) is compact and \( \mathcal{H}^m(\Gamma) < +\infty \), then for \( \mathcal{H}^m \)-a.e. \( x \in \Gamma \),

\[ \forall \delta \exists r \, \Sigma_n B(x,r) \subseteq C(x,N(x),r) \]

for some \( \dim_N x = m \).

Then \( \mathcal{H}^m \Sigma \) is \( \mathcal{P}_m \)-rectifiable.

---

**Remark.**

\[ \Sigma \cap B(x,r) \subseteq B(x,N(x),\omega(\delta)r) \]

means "near \( x \), points of \( \Sigma \) are even closer to \( xN(x) \)."

We also prove the converse and then

\[ \Sigma_{\lambda_i} : = \delta_{\lambda_i} (x^{-1} \Sigma) \xrightarrow{\lambda \to \infty} N(x) \]

---

**Theorem (A. - Merlo)**

\( \phi \) \( \mathcal{P}_m \)-r.m. on \( (\mathbb{G},d) \), \( N(x) \) complemented \( \phi \)-a.e. \( x \in \mathbb{G} \).

Then \( \exists \Gamma_i \), i.dif. graphs s.t. \( \phi(\mathbb{G} \setminus \bigcup \Gamma_i) = 0 \).
**Theorem [A.-Merlo]**

\( \phi \) \( P_m \)-rectifiable measure on \((G, d)\). \( W(x) \) complemented for \( \phi \)-a.e. \( x \in G \). Then

\[
\Theta^m(\phi, x) = \lim_{r \to 0} \frac{\phi(B(x, r))}{r^m}
\]

exists for \( \phi \)-a.e. \( x \in G \).

---

**Proof.**

1. \( \Gamma \) \((W, W)\)-Lipschitz graph with \( \text{Lip} \Gamma \leq \alpha \) and \( \mathcal{H}^m \Gamma \) is \( P_m \)-rectifiable. Then \( \mathcal{H}^m \Gamma \)-a.e. \( x \)

\[
C(\alpha) \leq \Theta^m(\mathcal{H}^m \Gamma, x) \leq 1
\]

with \( C(\alpha) \to 1 \) as \( \alpha \to 0 \)

[Idea: \( \Phi^*(\mathcal{H}^m \Gamma) \) mutually \( \ll \mathcal{H}^m \Gamma \) \( W \).

This allows to differentiate w.r.t. the family \( \mathcal{P}_W(\mathcal{B}(x, r) \Delta \Gamma): x \in \Gamma, r \text{ small} \).]

2. Reduce to countably many pieces \( 0 \) in \( \mathcal{D} \) by using \textit{Structure Results}
Theorem [A.-Merlo]

\[ \phi \text{ is } P_m^*- \text{rectifiable measure on } (G, \mu). \]
Assume that \( \phi \)-a.e. \( x \) all the possible tangents at \( x \) are complemented by at least one normal subgroup. Then

1. \( \phi \) is \( P_m^*- \text{rectifiable} \)
2. \( \exists F_i: A_i \subset W_i \rightarrow G \) Lipschitz with \( \dim_x W_i = m \) s.t.
   \[ \phi (G \setminus \bigcup_{i=1}^{\infty} F_i (A_i)) = 0 \]

Proof. [Very roughly]

1. **Rigidity**: At a point \( x \) all tangents have the same "stabilization vector" \( w = w_1 \oplus \ldots \oplus w_s \)
2. \( G = W_1 \oplus \ldots \oplus W_s \)
   \[ \text{if Normal, then Count!} \]
3. \( P_{\mu^i}: G \rightarrow W_i \) is metric Lipschitz homomorphism
**Def.** \( H^1 \equiv \mathbb{R}^3 \)

\[
\| (x_1, x_2, x_3) \|_H := \left( (x_1^2 + x_2^2 + x_3^2) \right)^{1/2}
\]

\[
d_H(x, y) := \| x^{-1} \cdot y \|_H
\]

**Thm. [A.-Merlo]**

Let \( \phi \) be R.m. on \( (H^1, d_H) \).

Assume \( 0 < \Theta^+(\phi, x) < +\infty \) \( \phi \)-a.e. \( x \in H^1 \).

Then \( \phi \ll \mathcal{H}^1 \)

\[\exists \psi_i : A_i \subseteq \mathbb{R} \to H^1 \text{ Lipschitz s.t.} \]

\[\phi \left( H^1 \setminus \bigcup_{i=1}^{+\infty} \psi_i(A_i) \right) = 0 \]

**Proof.**

1. \( \phi \)-a.e. \( x \in H^1 \) the tangent measures at \( x \) are \( \mathcal{H}^1 \)-Lipschitz for \( \theta > 0 \), \( \mathcal{H}^1 \)

1-dim hom. subgroup [Chousionis-Magnani-Tyson]

2. Any such \( \mathcal{H}^1 \) admit a normal subgroup in \( H^1 \).

Then apply Horsting-Mattila (consider)

In 3d see [Merlo, Merlo].
FURTHER STUDIES...

- Area formula [after Julia-Nicolussi-Golo-Vittone] for \( P_m \)-rectifiable measures with complemented tangents [A.-Merlo, WIP]
- Relations with Preiss' tangents [A.-Merlo, WIP]
- Corearea formulae
- Fine structure of \( P_m \)-rectifiable measures with non-complemented tangents.
THANK YOU FOR THE ATTENTION!